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Technical Note

A two-energy equation model for conduction and convection in porous media

Akira Nakayama *, Fujio Kuwahara, Masazumi Sugiyama, Guoliang Xu

Department of Mechanical Engineering, Shizuoka University, 351 Johoku, Hamamatsu 432-8561, Japan

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Abstract

A two-energy equation model is proposed for analyzing conduction and convection within porous media in local thermal non-equilibrium. Hsu's closure model for the tortuosity effect has been extended to treat not only conduction but also convection in porous media. The two energy equations for the individual phases are combined into a fourth-order ordinary differential equation, so as to treat one-dimensional steady-state problems. Then, the exact solutions are obtained for two fundamental cases, namely, one-dimensional steady conduction in a porous slab with internal heat generation within a solid, and also thermally developing unidirectional flow through a semi-infinite medium.

1. Introduction

In a fluid-saturated porous medium, the thermal diffusivity of the fluid phase may be much lower (or higher) than that of the solid structure. In transient heat conduction processes within such porous media, the assumption of local thermal equilibrium must be discarded, as pointed out by Kaviany [1], Hsu [2] and many others. Also, there are a number of "steady" situations in which the heat transfer process cannot be regarded as being in local thermal equilibrium. When there is a significant heat generation occurring in any one of two phases (either solid or fluid), the temperatures in the two phases are no longer equal [1,3]. The assumption of local thermal equilibrium cannot be used when we analyze the entrance region of packed column where a hot gas flows at a high speed. Numerous other physical situations, where local thermal equilibrium fails, can be found in Quintard [4] and Quintard and Whitaker [5].

Hsu [2] proposed a simplified two-energy equation model for transient heat conduction in porous media, and assessed the validity of the assumption of local thermal equilibrium. He follows a treatment, which is quite similar to that of Quintard and Whitaker [5], but mathematically more concise. The major difference between his form and the classical (heuristic) form is the appearance of additional coupling terms accounting for thermal tortuosity, which are related to the temperature gradient in the other phase. In this paper, we shall extend the closure model of Hsu [2], so as to treat not only conduction but also convection in porous media. Having established the macroscopic energy equations for both phases, useful exact solutions are obtained for two fundamental heat transfer processes associated with porous media, namely, steady conduction in a porous slab with internal heat generation within a solid, and also, thermally developing flow through a semi-infinite porous medium.

2. Volume-averaged energy equations and their closure modeling

Following Cheng [6] and Nakayama [7], the two-energy equation model proposed by Hsu [2] for the case of pure conduction in saturated porous media can be generalized to treat both conduction and convection in saturated porous media:

* Corresponding author. Fax: +81-53-478-1049.
E-mail address: tmanaka@ipc.shizuoka.ac.jp (A. Nakayama).

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Nomenclature

$a_{\rm sf}$	specific interfacial area
G	tortuosity parameter
$h_{\rm sf}$	interfacial convective heat transfer coefficient
Pe	Peclet number based on $\sqrt{k_{\rm f}/a_{\rm sf}h_{\rm sf}}$ and Darcian velocity
$Pe_{\rm d}$	Peclet number based on the particle diameter and Darcian velocity
X	dimensionless coordinate $X = x/\sqrt{k_{\rm f}/a_{\rm sf}h_{\rm sf}}$
ε	porosity
σ	thermal conductivity ratio of solid to fluid
Subscripts and superscripts	
f	fluid
s	solid
Special symbol	
$\langle\phi\rangle^{\rm f}$	intrinsic volume-average

$$\rho_{\rm f}C_{\rm pf}\left[\varepsilon\frac{\partial\langle T\rangle^{\rm f}}{\partial t} + \left\langle\vec{u}\right\rangle\cdot\nabla\langle T\rangle^{\rm f}\right] = \nabla\cdot\left[(\varepsilon+(1-\sigma)G)k_{\rm f}\nabla\langle T\rangle^{\rm f} + \bar{\bar{k}}_{\rm dis}\cdot\nabla\langle T\rangle^{\rm f}\right] + \left(a_{\rm sf}h_{\rm sf} - k_{\rm s}G\nabla^2\right)\left(\langle T\rangle^{\rm s} - \langle T\rangle^{\rm f}\right) + \varepsilon S_{\rm f}, \quad (1)$$

$$(1-\varepsilon)\rho_{\rm s}C_{\rm s}\frac{\partial\langle T\rangle^{\rm s}}{\partial t} = \nabla\cdot\left[(1-\varepsilon+(\sigma-1)G)k_{\rm s}\nabla\langle T\rangle^{\rm s}\right] - \left(a_{\rm sf}h_{\rm sf} - k_{\rm s}G\nabla^2\right)\left(\langle T\rangle^{\rm s} - \langle T\rangle^{\rm f}\right) + (1-\varepsilon)S_{\rm s}, \quad (2)$$

where $\langle T\rangle^{\rm f}$ and $\langle T\rangle^{\rm s}$ are the intrinsic volume-averaged temperature of the fluid phase and that of the solid phase, respectively. Moreover, ε is the porosity whereas $S_{\rm f}$ and $S_{\rm s}$ are the volumetric rates of heat generation for the fluid and solid phases, respectively. These volumetric heat generation rates and the material constants such as thermal conductivities $k_{\rm f,s}$, densities $\rho_{\rm f,s}$ and heat capacities $C_{\rm pf,s}$ are all assumed constant. The interfacial heat transfer coefficient $h_{\rm sf}$ and thermal dispersion tensor $\bar{\bar{k}}_{\rm dis}$ may be determined either empirically [3] or numerically [8–11]. Hsu [2] argues that the tortuosity parameter G must depend only on the local interfacial geometry and on the solid and fluid thermal properties, such that the expression for G obtained under the thermal equilibrium condition can be extended to the thermal non-equilibrium regime:

$$G = \frac{\left(k_{\rm stg}/k_{\rm f}\right) - \varepsilon - (1-\varepsilon)\sigma}{(\sigma-1)^2}. \quad (3)$$

In the foregoing expression, $\sigma = k_{\rm s}/k_{\rm f}$ is the thermal conductivity ratio of solid to fluid, whereas $k_{\rm stg}$ is the effective stagnant thermal conductivity of the saturated porous medium, which can be determined experimentally, or theoretically, using a structural model. As pointed out by Hsu, the parameter G is always negative such that the tortuosity effect is to reduce the effective thermal conductivity of the saturated porous medium from its upper limit $\varepsilon k_{\rm f} + (1-\varepsilon)k_{\rm s}$ based on the parallel model. The effect of parameter G on the stagnant thermal conductivity is discussed in detail in [2]. For the case of forced convection, however, the macroscopic conductivity due to thermal dispersion becomes much larger than the stagnant thermal conductivity, and controls the diffusion.

The two energy Eqs. (1) and (2) with the expressions for $h_{\rm sf}$, $\bar{\bar{k}}_{\rm dis}$ and $k_{\rm stg}$ (given either empirically or theoretically), constitute a closed set of the two-energy equation model for conduction and convection in fluid-saturated porous media. Unlike some existing two-equation models that contain a number of unknown transfer coefficients, the present model is quite concise such that it can readily be employed to investigate a number of steady and unsteady transport processes in porous media, numerically or analytically.

For sample calculations, we may use Wakao and Kaguei's empirical expressions for $h_{\rm sf}$ and $\bar{\bar{k}}_{\rm dis}$ [3] and Hsu's analytical expression based on a three-dimensional cube model for $k_{\rm stg}$ [12]:

$$\frac{h_{\rm sf}d_{\rm p}}{k_{\rm f}} = 2 + 1.1Pr_{\rm f}^{1/3}\left(\frac{\rho_{\rm f}\left|\left\langle\vec{u}\right\rangle\right|d_{\rm p}}{\mu_{\rm f}}\right)^{0.6}, \quad (4a)$$

$$\frac{\left(\bar{\bar{k}}_{\rm dis}\right)_{xx}}{k_{\rm f}} = 0.5\frac{\rho_{\rm f}C_{\rm pf}\left|\left\langle\vec{u}\right\rangle\right|d_{\rm p}}{k_{\rm f}}, \quad (4b)$$

$$\frac{k_{\rm stg}}{k_{\rm f}} = 1 - (1-\varepsilon)^{2/3} + \frac{(1-\varepsilon)^{2/3}\sigma}{\left(1-(1-\varepsilon)^{1/3}\right)\sigma + (1-\varepsilon)^{1/3}}, \quad (4c)$$

where $d_{\rm p}$ is the particle diameter. Eqs. (4a) and (4b) suggest that both thermal dispersion and interfacial heat transfer become significant as we increase the Reynolds number.

3. One-dimensional steady problems

In order to elucidate the relevancy of the treatments based on the present two-energy equation model, we shall consider one-dimensional steady problems and derive a fourth-order ordinary differential equation, which can easily be integrated to find the temperature fields in both phases.

3.1. Governing equations for one-dimensional steady problems

For the one-dimensional steady problems, Eqs. (1) and (2) can be combined to eliminate $\langle T\rangle^{s}$. After considerable manipulation, we obtain the following fourth-order O.D.E. for determining $\langle T\rangle^{f}$:

$$\beta\frac{d^4\langle T\rangle^f}{dX^4}-\frac{d^2\langle T\rangle^f}{dX^2}-Pe\Big/\left(\frac{k_{stg}+\left(\bar{\bar{k}}_{dis}\right)_{xx}}{k_f}\right)\times\left(\sigma(1-\varepsilon+\sigma G)\frac{d^3\langle T\rangle^f}{dX^3}-\frac{d\langle T\rangle^f}{dX}\right)=(\varepsilon S_f+(1-\varepsilon)S_s)\Big/a_{sf}h_{sf}\left(\frac{k_{stg}+\left(\bar{\bar{k}}_{dis}\right)_{xx}}{k_f}\right), \quad (5)$$

where the dimensionless quantities are given by

$$X=x\Big/\sqrt{\frac{k_f}{a_{sf}h_{sf}}}, \quad (6a)$$

$$Pe=\frac{\rho_f C_{pf}\left|\left\langle\vec{u}\right\rangle\right|}{k_f}\sqrt{\frac{k_f}{a_{sf}h_{sf}}} \quad (6b)$$

and

$$\beta=\frac{\sigma((1-\varepsilon)(\varepsilon+G)+\varepsilon\sigma G+(1-\varepsilon+\sigma G)(\bar{\bar{k}}_{dis})_{xx}/k_f)}{(k_{stg}+(\bar{\bar{k}}_{dis})_{xx})/k_f}. \quad (6c)$$

Once $\langle T\rangle^{f}$ is known, $\langle T\rangle^{s}$ can readily be determined from

$$\langle T\rangle^s=\langle T\rangle^f-\frac{\beta}{\sigma(1-\varepsilon+(\sigma-1)G)}\left(\frac{k_{stg}+\left(\bar{\bar{k}}_{dis}\right)_{xx}}{k_f}\right)\times\frac{d^2\langle T\rangle^f}{dX^2}+\frac{1-\varepsilon+\sigma G}{1-\varepsilon+(\sigma-1)G}Pe\frac{d\langle T\rangle^f}{dX}-\frac{G}{1-\varepsilon+(\sigma-1)G}\left(\frac{\varepsilon S_f+(1-\varepsilon)S_s}{a_{sf}h_{sf}}\right)-\frac{\varepsilon S_f}{a_{sf}h_{sf}}. \quad (7)$$

We shall now consider two fundamental cases in which interesting analytical solutions exist.

3.2. Heat conduction with heat generation in the solid phase

Let us consider steady heat conduction through a porous slab saturated with a fluid, as shown in Fig. 1. Heat generation takes place within the solid phase, such

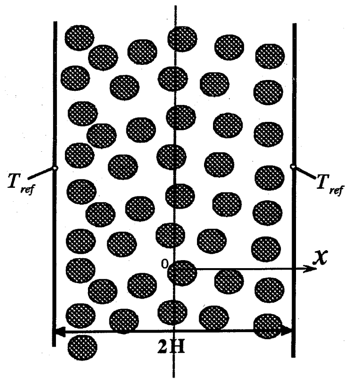

Fig. 1. Porous slab with heat generation in the solid phase.

that $S_f=0$ and $S_s=\text{const}\neq 0$. For this case, Eqs. (5) and (7) reduce to

$$\beta\frac{d^4\langle T\rangle^f}{dX^4}-\frac{d^2\langle T\rangle^f}{dX^2}=\frac{(1-\varepsilon)S_s}{a_{sf}h_{sf}\left(k_{stg}/k_f\right)}, \quad (8)$$

$$\langle T\rangle^s=\langle T\rangle^f-\frac{\beta}{\sigma(1-\varepsilon+(\sigma-1)G)}\left(\frac{k_{stg}}{k_f}\right)\frac{d^2\langle T\rangle^f}{dX^2}-\frac{G}{1-\varepsilon+(\sigma-1)G}\left(\frac{(1-\varepsilon)S_s}{a_{sf}h_{sf}}\right), \quad (9)$$

where

$$\beta=\frac{\sigma((1-\varepsilon)(\varepsilon+G)+\varepsilon\sigma G)}{\left(k_{stg}/k_f\right)}. \quad (10)$$

The boundary conditions are given by

$$X=0:\quad \frac{d\langle T\rangle^f}{dX}=\frac{d^3\langle T\rangle^f}{dX^3}=0, \quad (11a)$$

$$X=\frac{H}{\sqrt{k_f/a_{sf}h_{sf}}}:\quad \langle T\rangle^f=\langle T\rangle^s=T_{ref}. \quad (11b)$$

Hence, the solutions are given by

$$\left(\langle T\rangle^f-T_{ref}\right)\Big/\left(\frac{(1-\varepsilon)S_s}{a_{sf}h_{sf}\left(k_{stg}/k_f\right)}\right)=(\beta-\sigma G)\left(\frac{\cosh\left(X/\sqrt{\beta}\right)}{\cosh\left(H\Big/\sqrt{\beta k_f/a_{sf}h_{sf}}\right)}-1\right)-\frac{1}{2}\left(X^2-\left(\frac{H}{\sqrt{k_f/a_{sf}h_{sf}}}\right)^2\right), \quad (12)$$

$$
\left(\langle T\rangle^{\mathrm{s}}-T_{\mathrm{ref}}\right)\Big/\left(\frac{(1-\varepsilon)S_{\mathrm{s}}}{a_{\mathrm{sf}}h_{\mathrm{sf}}\left(k_{\mathrm{stg}}/k_{\mathrm{f}}\right)}\right)
= (\beta-\sigma G)\left(1-\frac{\left(k_{\mathrm{stg}}/k_{\mathrm{f}}\right)}{\sigma(1-\varepsilon+(\sigma-1)G)}\right)
\times\left(\frac{\cosh\left(X/\sqrt{\beta}\right)}{\cosh\left(H\Big/\sqrt{\beta k_{\mathrm{f}}/a_{\mathrm{sf}}h_{\mathrm{sf}}}\right)}-1\right)
-\frac{1}{2}\left(X^{2}-\left(\frac{H}{\sqrt{k_{\mathrm{f}}/a_{\mathrm{sf}}h_{\mathrm{sf}}}}\right)^{2}\right). \tag{13}
$$

The fluid and solid temperature distributions with $\varepsilon = 0.4$ and $H = 5\sqrt{k_{\mathrm{f}}/a_{\mathrm{sf}}h_{\mathrm{sf}}}$ are presented in Fig. 2 for $\sigma = 40$. The figure clearly shows that the heat generated in the solid phase constantly transfers into the fluid phase, such that the local thermal equilibrium assumption may not be valid for the case of large σ and small H.

3.3. Thermally developing unidirectional flow through a semi-infinite porous medium

For this second fundamental case, a hot fluid enters into a semi-infinite porous medium at the velocity $|\langle\vec{u}\rangle|$, as shown in Fig. 3. The exhaust gas flow through a catalytic converter may be modeled in this fashion as in the study of diffusion-controlled catalytic reaction [13].

As the fluid goes downstream, the thermal equilibrium state will be attained asymptotically. Since there is no internal heat generation, we have $S_{\mathrm{f}} = S_{\mathrm{s}} = 0$. Hence, Eqs. (5) and (7) reduce to

$$
\beta\frac{\mathrm{d}^{4}\langle T\rangle^{\mathrm{f}}}{\mathrm{d}X^{4}}-\frac{\mathrm{d}^{2}\langle T\rangle^{\mathrm{f}}}{\mathrm{d}X^{2}}-\left\{Pe\Big/\left(\frac{k_{\mathrm{stg}}+\left(\bar{\bar{k}}_{\mathrm{dis}}\right)_{xx}}{k_{\mathrm{f}}}\right)\right\}
\times\left(\sigma(1-\varepsilon+\sigma G)\frac{\mathrm{d}^{3}\langle T\rangle^{\mathrm{f}}}{\mathrm{d}X^{3}}-\frac{\mathrm{d}\langle T\rangle^{\mathrm{f}}}{\mathrm{d}X}\right)=0, \tag{14}
$$

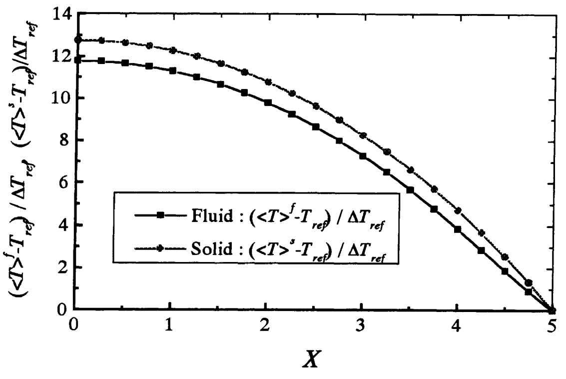

Fig. 2. Fluid and solid temperature distributions within a porous slab $\varepsilon = 0.4, \sigma = 40$ and $H = 5\sqrt{k_{\mathrm{f}}/a_{\mathrm{sf}}h_{\mathrm{sf}}}$).

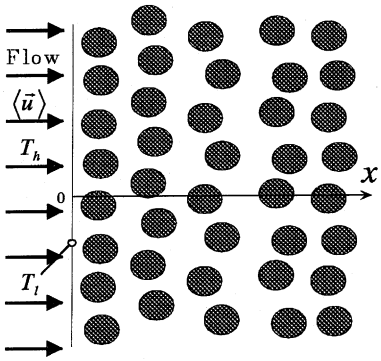

Fig. 3. Hot fluid passing through a semi-infinite porous medium.

$$
\langle T\rangle^{\mathrm{s}}=\langle T\rangle^{\mathrm{f}}-\frac{\beta}{\sigma(1-\varepsilon+(\sigma-1)G)}\left(\frac{k_{\mathrm{stg}}+\left(\bar{\bar{k}}_{\mathrm{dis}}\right)_{xx}}{k_{\mathrm{f}}}\right)
\times\frac{\mathrm{d}^{2}\langle T\rangle^{\mathrm{f}}}{\mathrm{d}X^{2}}+\frac{1-\varepsilon+\sigma G}{1-\varepsilon+(\sigma-1)G}Pe\frac{\mathrm{d}\langle T\rangle^{\mathrm{f}}}{\mathrm{d}X}, \tag{15}
$$

where β is given by (6c). The boundary conditions for the case are given by

$$
X=0:\qquad \langle T\rangle^{\mathrm{f}}=T_{\mathrm{h}},\quad \langle T\rangle^{\mathrm{s}}=T_{\mathrm{l}}, \tag{16a}
$$

$$
X=\infty:\qquad \langle T\rangle^{\mathrm{f}}=\langle T\rangle^{\mathrm{s}},\quad \frac{\mathrm{d}\langle T\rangle^{\mathrm{f}}}{\mathrm{d}X}=0. \tag{16b}
$$

The solutions are given by

$$
\frac{\langle T\rangle^{\mathrm{f}}-T_{\mathrm{l}}}{(T_{\mathrm{h}}-T_{\mathrm{l}})}
=1+\left\{\gamma\sigma(1-\varepsilon+(\sigma-1)G)\Big/\left[Pe+\gamma\left(\frac{k_{\mathrm{stg}}+\left(\bar{\bar{k}}_{\mathrm{dis}}\right)_{xx}}{k_{\mathrm{f}}}\right)\right]\right\}
\times(\exp(-\gamma X)-1), \tag{17}
$$

$$
\frac{\langle T\rangle^{\mathrm{s}}-T_{\mathrm{l}}}{(T_{\mathrm{h}}-T_{\mathrm{l}})}
=1+\left\{\gamma\sigma(1-\varepsilon+(\sigma-1)G)\Big/\left[Pe+\gamma\left(\frac{k_{\mathrm{stg}}+\left(\bar{\bar{k}}_{\mathrm{dis}}\right)_{xx}}{k_{\mathrm{f}}}\right)\right]\right\}
\times(\exp(-\gamma X)-1)-\exp(-\gamma X), \tag{18}
$$

where $-\gamma$ is the negative real root which can be uniquely determined from the following cubic characteristic equation

$$
\beta(-\gamma)^{3}-\left\{Pe\sigma(1-\varepsilon+\sigma G)\Big/\left(\frac{k_{\mathrm{stg}}+\left(\bar{\bar{k}}_{\mathrm{dis}}\right)_{xx}}{k_{\mathrm{f}}}\right)\right\}
\times(-\gamma)^{2}-(-\gamma)+\left\{Pe\Big/\left(\frac{k_{\mathrm{stg}}+\left(\bar{\bar{k}}_{\mathrm{dis}}\right)_{xx}}{k_{\mathrm{f}}}\right)\right\}=0. \tag{19}
$$

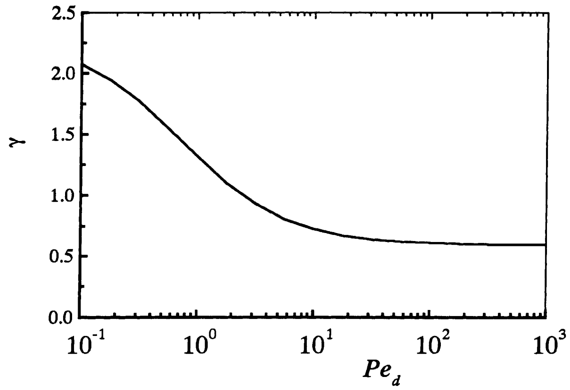

Fig. 4. Effect of particle Peclet number on γ.

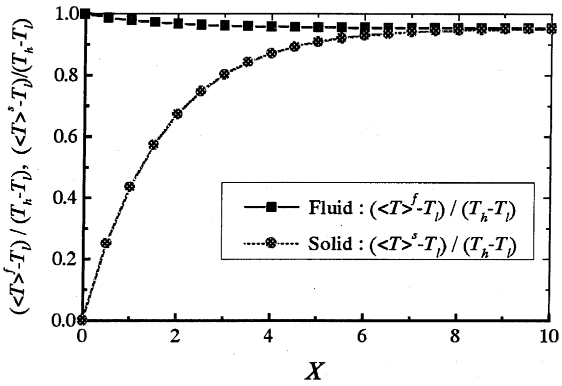

Fig. 5. Fluid and solid temperature distributions in a semi-infinite medium ($\varepsilon = 0.4, \sigma = 40, Pr_f = 1$ and $Pe_d = 100$).

Fig. 4 shows the effect of the particle Peclet number Pe_d on the root γ for the case of $Pr_f = 1, \varepsilon = 0.4$ and $\sigma = 40$. Note that the Peclet number Pe is related to the particle Peclet number via (4a)

$$Pe = \frac{Pe_d}{\left(6(1-\varepsilon)\left(2 + 1.1 Pr_f^{-4/15} Pe_d^{0.6}\right)\right)^{0.5}}, \tag{20a}$$

where

$$Pe_d = \frac{\rho_f C_{pf} \left|\left\langle \vec{u} \right\rangle\right| d_p}{k_f}. \tag{20b}$$

The temperature distributions for both phases for the case of $\varepsilon = 0.4$, $\sigma = 40, Pr_f = 1$ and $Pe_d = 100$ are plotted in Fig. 5. The fluid is cooled down to the thermal equilibrium (asymptotic) temperature, as it flows downstream, releasing its heat to the solid phase.

4. Conclusions

A two-energy equation model was established to analyze both conduction and convection phenomena in fluid-saturated porous media. It has been shown that, for one-dimensional steady problems, the two energy equations can be combined to form a fourth-order O.D.E. with respect to the intrinsically averaged fluid temperature. The resulting O.D.E. were solved with appropriate boundary conditions to find exact solutions for two fundamental problems, namely, one-dimensional steady conduction in a porous slab with internal heat generation within a solid, and also, thermally developing unidirectional flow through a semi-infinite medium. Experimental validation of the present two-energy equation model may not be an easy task since careful temperature measurements for the individual phases are required.

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